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Critical Exponents for the Contact Process under the Triangle Condition

David J. Barsky[1] and C. Chris Wu[2]

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We show that a continuous-time version of the triangle condition for percolation implies mean-field values for several contact process critical exponents. Our results support the belief that the upper critical (spatial) dimension for the contact process is four.

KEY WORDS: Contact process; triangle condition; critical exponent; upper critical dimension.

1. INTRODUCTION

The contact process is an interacting particle system which, due to its *percolation substructure*, often becomes tractable when studied using percolation-theoretic techniques (see [D2], [G1], [H2], and [L]). The contact process can be regarded as modeling the spread of an epidemic through a population; although the process itself is dynamic, its entire history is a static (oriented) percolation model. Sites in some underlying lattice heal and are infected by sick neighbors independently; when the ratio of infection rate to healing rate is small (subcritical), the radius of some initially finite infection has an exponentially decaying tail, whereas when the ratio is large (supercritical), every initial infection has a positive probability of persisting forever. It is of particular interest to study the behavior of various quantities in the vicinity of the critical point which separates these two regimes. These quantities (one of which is the probability of infinite persistence of an infection starting at a single site) are expected

[1] California State University San Marcos, San Marcos, California 92096; e-mail: djbarsky@csusm.edu.

[2] Department of Mathematics, Penn State University, Beaver Campus, Monaca, Pennsylvania 15061; e-mail: wu@math.psu.edu.

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to display power law behavior as the ratio parameter (or some other parameter) is varied near its critical value. The exponents in these power laws are called *critical exponents*, and they are predicted by *universality* arguments to depend only on the lattice dimension. It is further believed that above a certain *upper critical dimension*, the critical exponents also lose their dimension dependence, and simply assume their mean-field values.

We prove that when a contact process analogue of the Barsky-Aizenman triangle condition for percolation is satisfied, several critical exponents (β, γ, δ) take on their mean-field values. Our work is based on the analysis of [BA] and [AN]. A triangle condition for (unoriented) percolation models was first introduced in [AN], where a pair of complementary differential inequalities were obtained to prove that this condition implied the mean-field value for γ. Nguyen [Ng] further showed that the gap exponents were mean-field under the same condition. An extension of this condition was proven in [BA] in the context of POP (partially oriented percolation) models to lead to a differential inequality complementary to one in [AB], with the result that β and δ were also mean field. For unoriented percolation, the triangle condition was verified by Hara and Slade [HS] in high dimensions—in the case of a spread-out model, this meaning $d > 6$, by using the lace expansion method. Using the same method, Nguyen and Yang [NY] verified the triangle condition for an oriented percolation model, but in this case the condition is already met for a (spread-out) model on $\mathbb{Z}^d$ with $d > 4$. To complete the picture for the contact process on $\mathbb{Z}^d$, one needs to do two things: (1) prove that the triangle condition for the contact process implies the mean-field behavior (the analogue of the diagrammatic inequality analysis of [AN] and [BA]), and (2) verify the triangle condition for the contact process (the analogue of the lace expansion analysis of [HS] and [NY]). This paper addresses problem (1). Problem (2) is still open; a complete extension of the machinery of [HS] and [NY] to the contact process has not yet been made, although it is generally believed (see [NY]) that the triangle condition for the contact process should also be satisfied when $d > 4$. Our results together with this belief imply that the critical exponents for the contact process assume their mean-field values when $d > 4$. The class of models we consider includes both the *basic* (or nearest-neighbor) contact process and spread-out processes. Our arguments are in a form which allows for their extension to the case of infections of unbounded range with a minimal amount of difficulty, although our focus is on finite-range models. Our results are valid for the contact process on any vertex-transitive lattice, and probably the vertex-transitivity requirement is not crucial. For nearest-neighbor contact processes on trees [P], it was shown in [W] that the triangle condition is satisfied when the degree of the tree exceeds 4. This last result has recently been extended in [S] to the case of trees with degree 2 or more—a tree of degree 1 being the line.

Our basic strategy is to look for differential inequalities that can be integrated to show that the behavior of some observable in the vicinity of the critical ratio is always bounded by the behavior of its mean field approximation. Our two fundamental differential inequalities are the contact process analogues of percolation inequalities from [AN] (used in the analysis of γ) and [BA] (used for β and δ). We prove the contact process version of the first of these, the second already being done in [BG2]. Both inequalities are obtained by neglecting certain overlapping routes of infection. We estimate this effect and derive complementary inequalities (by an inclusion-exclusion argument, called *factorization* in [BA]); under the triangle condition, the correction terms can be controlled through a *delocalization* argument.

We obtain the differential inequalities in finite space-time volumes by discretizing the contact process and then reworking percolation arguments in the spirit of [AN] and [BA]. We comment here on how our work differs from these papers, and on why it is not possible to directly apply their results to an oriented percolation model and then simply pass to the limit to get the corresponding results for the contact process. First, although the discretization of any finite-range contact process is a "well-connected" POP model in the sense of [BA], a naive application of the delocalization procedure(s) of ([AN] and) [BA] leads to complementary inequalities that become meaningless in the continuum limit (because the factor multiplying the lower bound goes to zero). We present a new delocalization argument, taking advantage of the Markov property of the contact process, that not only avoids this problem, but which also allows us to treat together the finite- and long-range cases—which were separately handled by different arguments in [BA]. (In [AN], delocalizing was discussed only for finite-range models.) Second, in the factorization step, in order to get the correct scaling for the continuum limit to work, we must use estimates on the probabilities of infection passing through specified sets that are more detailed than those of standard percolation; this introduces some lower order diagrams (such as the bubble) which are bounded by the triangle, but which do not explicitly appear in [AN] or [BA]. Finally, instead of separate ad-hoc arguments for the two pairs of differential inequalities (one for β and δ, and another for γ), we provide a single unified framework for delocalizing and factorizing that applies equally well to both situations, thus emphasizing that the derivations of these inequalities are largely *the same*, and not just roughly analogous.

In Section 2, we define the contact process, give precise statements of our results, and introduce the finite volume and discretized models. We integrate the finite-volume differential inequalities to get the power laws in Section 3. The general delocalization and factorization arguments are given in Section 4 (with one proof deferred to an Appendix), and in Section 5 they are specialized (under the triangle condition) to yield the differential inequalities used in Section 3.

2. THE SET-UP

2.1. The Continuous-Time Model

We define here the contact process on $\mathbb{Z}^d$, the d-dimensional integer lattice. We use the graphical representation of Harris [H1], [H2] (see also [D2], [G1], [G2] and [L]), in which we first form a larger graph $\mathbb{L}$ by taking the Cartesian product of the space $\mathbb{Z}^d$ with a time-axis: $\mathbb{L} = \mathbb{Z}^d \times [0, \infty)$. Along each time-line $\{x\} \times [0, \infty)$ (which we regard as being *vertical*), let there be an independent Poisson distribution (with density 1) of points which mark deaths of the infection. For each ordered pair of time-lines, say $\{x\} \times [0, \infty)$ and $\{y\} \times [0, \infty)$, there is an independent Poisson process (with density λJ_{xy}), that serves to mark when an infection can spread from x (if it is currently infected) to y; an arrow is drawn from $X = (x, t)$ to $Y = (y, t)$ for every point t of this process. Our convention is to use capital letters for points in $\mathbb{L}$, and lower case letters for their spatial and temporal components.

We require the parameters J_{xy} to be nonnegative, translation invariant ($J_{x+z,\, y+z} = J_{xy}$ for all x, y and z in $\mathbb{Z}^d$), and such that $0 < |J| = \sum_{y \in \mathbb{Z}^d} J_{xy}$. We consider the finite-range case, where there is an R such that $J_{xy} = 0$ whenever $|x - y| \geqslant R$ (take $|\cdot|$ to be the ℓ_1-norm on $\mathbb{Z}^d$: $|x| = |x_1| + \cdots + |x_n|$ for $x = (x_1, ..., x_n)$); if one were to use our arguments for infinite-range models, one would have to require that $|J| < \infty$. For simplicity, we also assume that $J_{xy} = J_{yx}$, although this requirement could be relaxed at the expense of a slightly more complicated construction in Section 4.

We say that y is a neighbor of x if J_{xy} is positive, and extend this notion to points in space-time by saying that $Y = (y, t)$ is a neighbor of $X = (x, t)$ when y is a neighbor of x. For the basic contact process, J_{xy} is 1 if $|x - y| = 1$, and 0 otherwise. For a contact process similar to the spread-out oriented percolation model of [NY] (see also [HS]), one requires J_{xy} to be exponentially decaying in $|x - y|$. To simplify the notation, we fix the parameters J_{xy} beforehand (which amounts to fixing the set of neighbors, and the relative infection rates), and only vary λ over $(0, \infty)$. We write P^λ and E^λ for the probability measure and expectation operator arising from the product of the independent death and arrow processes.

A point (x, t_1) in $\mathbb{L}$ is said to be *connected* to another point (y, t_2) if and only if it is possible to trace a path from (x, t_1) to (y, t_2) that only uses arrows (in the direction that they point) from time-line to time-line and vertical segments along time-lines between deaths—with those segments being traversed in the upward direction; note that such a connection can only occur if $t_2 \geqslant t_1$. We write "$(x, t_1) \to (y, t_2)$" to denote the event that (x, t_1) is connected to (y, t_2); more generally "$(x, t_1) \to D$" means that (x, t_1) is connected to some point in $D \subset \mathbb{L}$. Define the *cluster* of (x, t) to be the set of all points in $\mathbb{L}$ to which (x, t) is connected: $C(x, t) = \{(y, t') : (x, t) \to (y, t')\}$. In the special case of the space-time origin, $\mathbf{O}$, we write C for $C(\mathbf{O})$.

One of the important quantities in the study of the contact process is the *survival probability*

$$\theta(\lambda) = P^{\lambda}(C \cap (\mathbb{Z}^d \times \{t\}) \neq \varnothing \text{ for all } t) \tag{2.1}$$

The critical point is defined to be

$$\lambda_c = \sup\{\lambda : \theta(\lambda) = 0\} \tag{2.2}$$

it is well – known that $0 < \lambda_c < \infty$. Another key quantity is the expected total duration of the infection

$$\chi(\lambda) = E^{\lambda}(\|C\|) = E^{\lambda}\left(\int_0^{\infty} \text{card}(C \cap (\mathbb{Z}^d \times \{t\})\, dt\right) \tag{2.3}$$

where $\|C\|$ is the (one-dimensional) Lebesgue measure of C (in space-time), and card(A) is the number of points in A. It is a fundamental result of [BG2] (see also [A]) that additionally

$$\lambda_c = \sup\{\lambda : \chi(\lambda) < \infty\} \tag{2.4}$$

The exponents β and γ characterize the behavior of θ and χ as the critical point is approached, and δ describes the decay of the size of the critical cluster:

$$\theta(\lambda) \sim (\lambda - \lambda_c)^{\beta} \qquad \text{as} \quad \lambda \searrow \lambda_c \tag{2.5}$$

$$\chi(\lambda) \sim (\lambda_c - \lambda)^{-\gamma} \qquad \text{as} \quad \lambda \nearrow \lambda_c \tag{2.6}$$

and

$$P^{\lambda_c}(\|C\| \geqslant s) \sim s^{-1/\delta} \qquad \text{as} \quad s \to \infty \tag{2.7}$$

We prefer to describe δ in terms of the Laplace transform of the (critical) distribution of $\|C\|$:

$$1 - E^{\lambda_c}(e^{-h\,\|C\|}) \sim h^{1/\delta} \qquad \text{as} \quad h \searrow 0 \tag{2.8}$$

the correspondence between (2.7) and (2.8) may be made via a Tauberian theorem (see, e.g., [F]).

A convenient interpretation for the quantity in (2.8) is obtained (following [AB], [BA]) by turning the parameter h into a ghost field. Specifically, we put an independent Poisson distribution (with density h) of green points (or ghost sites) along each timeline, and denote their entire collection G. Writing

$$M(\lambda, h) = P^{\lambda, h}(\mathbf{O} \to G) = P^{\lambda, h}(C \cap G \neq \varnothing) = 1 - E^{\lambda}(e^{-h\,\|C\|}) \tag{2.9}$$

(where the last equality is most easily seen by conditioning on $\|C\|$), we have by the Dominated Convergence Theorem that

$$\lim_{h \searrow 0} M(\lambda, h) = 1 - \lim_{h \searrow 0} E^{\lambda}(e^{-h\,\|C\|}) = 1 - P^{\lambda}(\|C\| < \infty) = \theta(\lambda) \tag{2.10}$$

An important related quantity is

$$\chi(\lambda, h) = \frac{\partial}{\partial h} M(\lambda, h) = E^{\lambda}(\|C\|\, e^{-h\,\|C\|}) \tag{2.11}$$

whenever $\theta(\lambda) = 0$,

$$\lim_{h \searrow 0} \chi(\lambda, h) = \chi(\lambda) \tag{2.12}$$

2.2. Statement of Results

Our triangle condition is that

$$\lim_{r \to \infty} \nabla(\lambda_c; r) = 0 \tag{2.13}$$

where $\nabla(\lambda; r)$ is a triangle diagram function with one vertex opened (spatially) by at least distance r:

$$\nabla(\lambda; r) = \sup_{\substack{z : |z| \geqslant r \\ s \geqslant 0}} \sum_{u, v} \int_s^{\infty} dt_1 \int_{t_1}^{\infty} dt_2\, P^{\lambda}(\mathbf{O} \to (v, t_2)) \times P^{\lambda}((z, s) \to (u, t_1))\, P^{\lambda}((u, t_1) \to (v, t_2)) \tag{2.14}$$

Condition (2.13) is a rather natural extension of the Barsky-Aizenman triangle condition for (discrete) POP models to the (continuous-time) contact process; note that only the spatial coordinates of **O** and (z, s) are required to be widely separated.

We shall show that when the triangle condition (2.13) is met, then there exist positive constants $c_1,..., c_6$ such that for λ sufficiently close to λ_c and (positive) h sufficiently close to 0,

$$c_1(\lambda - \lambda_c) \leqslant \theta(\lambda) \underset{(\nabla)}{\leqslant} c_2(\lambda - \lambda_c) \qquad \text{for} \quad \lambda \geqslant \lambda_c, \tag{2.15}$$

$$c_3(\lambda_c - \lambda)^{-1} \leqslant \chi(\lambda) \underset{(\nabla)}{\leqslant} c_4(\lambda_c - \lambda)^{-1} \qquad \text{for} \quad \lambda < \lambda_c \tag{2.16}$$

and

$$c_5 h^{1/2} \leqslant M(\lambda_c, h) \underset{(\nabla)}{\leqslant} c_6 h^{1/2} \qquad \text{for} \quad h > 0 \tag{2.17}$$

Remarks. 1. The lower bounds in (2.15) and (2.17) were already established for the contact process in [BG2] (see also [A]); we shall prove the remaining four inequalities, with the upper bounds (underset with a ∇) being obtained under the triangle condition.

2. One consequence of the upper bound in (2.15) (or in (2.17)—see (2.10)) is that finite infections die out w.p.1 at the critical point: $\theta(\lambda_c) = 0$. However, this result has already been shown to hold for the contact process on $\mathbb{Z}^d$ in [BG1] and [BGr], and on trees in [P] and [MSZ]—without making any use of the triangle condition, upon which our argument depends.

3. In proving the lower bound in (2.16), we demonstrate the divergence of $\chi(\lambda)$ as $\lambda \nearrow \lambda_c$. By the monotonicity of χ in λ, it then also follows that $\chi(\lambda_c) = \infty$.

We restate inequalities (2.15)–(2.17) in terms of their implications for the critical exponents, and in the order in which they will be proved.

Theorem 1. When the triangle condition (2.13) is satisfied, then

(a) $\delta = 2$—in the sense that (2.17) is satisfied,

(b) $\beta = 1$—in the sense that (2.15) is satisfied, and

(c) $\gamma = 1$—in the sense that (2.16) is satisfied.

In proving the right-hand sides of (2.15)–(2.17) we need to control, in addition to the triangle, a pair of lower order diagrams: the bubble and the arc:

$$B(\lambda; r) = \sup_{\substack{z:|z|\geqslant r\\ s\geqslant 0}} \sum_u \int_s^\infty dt\, P^\lambda(\mathbf{O} \to (u, t))\, P^\lambda((z, s)) \to (u, t)) \tag{2.18}$$

and

$$A(\lambda; r) = \sup_{\substack{z:|z|\geqslant r\\ s\geqslant 0}} P^\lambda(\mathbf{O} \to (z, s)) \tag{2.19}$$

It is obvious in ordinary percolation and Ising models that the arc is no larger than the bubble, which is no larger than the triangle. We quickly outline a proof of related inequalities for the contact process:

$$A(\lambda; r) \leqslant 2B(\lambda; r) \leqslant 4\, \nabla(\lambda; r) \tag{2.20}$$

Note that "$(z, s) \to (z, t)$" occurs if there are no deaths along the z time-line between times s and t—which happens with probability $e^{-(t-s)}$. By the Markov property of the process, we thus have

$$P^\lambda(\mathbf{O} \to (z, t)) \geqslant P^\lambda(\mathbf{O} \to (z, s)) \cdot e^{-(t-s)} \tag{2.21}$$

hence

$$\int_s^\infty dt\, P^\lambda(\mathbf{O} \to (z, t))\, P^\lambda((z, s)) \to (z, t))$$

$$\geqslant P^\lambda(\mathbf{O} \to (z, s)) \int_s^\infty dt\, e^{-2(t-s)} = \tfrac{1}{2} P^\lambda(\mathbf{O} \to (z, s)) \tag{2.22}$$

The first inequality in (2.20) follows immediately from (2.22), and the second has a similar derivation.

2.3. The Finite-Volume Model

For $L > 0$, let $\hat{\Lambda}_L = [-L, L]^d \cap \mathbb{Z}^d$ and write $\Lambda_L = \hat{\Lambda}_L \times [0, L]$. We build the contact process on $\hat{\Lambda}_L$ (up to time L) by the graphical construction of Subsection 2.1. For technical reasons, we choose to use *free* boundary conditions in both space and time, i.e., we do not allow the connecting paths to use arrows terminating or originating on a time-line $\{x\} \times [0, \infty)$

with $x \notin [-L, L]^d$, nor may they cross $\mathbb{Z}^d \times \{L\}$ along the timelines. (The choice of temporally free but spatially *periodic* boundary conditions would slightly simplify some arguments in Section 5, but only at the cost of other inconveniences [BW].) Write P_L^λ (resp., $P_L^{\lambda, h}$) for the measure corresponding to the arrow and death (resp., arrow, death and green) process in the finite space-time volume Λ_L, and denote the associated expectation operator by E_L^λ.

The finite-volume quantities $M_L(\lambda, h)$, $\chi_L(\lambda, h)$, $\chi_L(\lambda)$, $\nabla_L(\lambda, r)$, $B_L(\lambda, r)$ and $A_L(\lambda, r)$ are defined by (2.9), (2.11), (2.3), (2.14), (2.18) and (2.19), resp., with the changes that C is replaced by C_L (the cluster of the origin in Λ_L), G is replaced by G_L (the set of green points in Λ_L), infinite-volume probabilities and expectations are replaced by their finite-volume counterparts, sums are restricted to sites in $\hat{\Lambda}_L$, and integrals are restricted to times in $[0, L]$.

The natural coupling between the finite- and infinite-volume processes, shows that $C_L \nearrow C$ as $L \to \infty$, and so, by the Monotone Convergence Theorem, $M_L(\lambda, h) \to M(\lambda, h)$ and $\chi_L(\lambda) \to \chi(\lambda)$ in the infinite-volume limit. It is also convenient to introduce another finite-volume quantity related to $\chi_L(\lambda)$:

$$\bar{\chi}_L(\lambda) = \max_{z \in \hat{\Lambda}_L} \sum_{y \in \hat{\Lambda}_L} \int_0^L P_L^\lambda((z, 0) \to (y, t))\, dt \tag{2.23}$$

Note that

$$\lim_{L \to \infty} \bar{\chi}_L(\lambda) = \chi(\lambda) \tag{2.24}$$

since for each L we have $\chi_L(\lambda) \leqslant \bar{\chi}_L(\lambda) \leqslant \chi(\lambda)$. The only other fact regarding the comparison between finite- and infinite-volume processes is the trivial (in light of the monotonicity in L of two-point connection probabilities) observation that the finite-volume arc, bubble and triangle diagrams are all bounded above by the corresponding infinite-volume diagrams.

2.4. The Discrete-Time Model

Here we discretize the contact process (in the finite volume Λ_L) to obtain an oriented percolation model on the graph $\Lambda_{L,\varepsilon} = \hat{\Lambda}_L \times (\varepsilon\mathbb{Z} \cap [0, L])$. Connections in this model are made along (paths of) open oriented bonds, which come in two types: the *horizontal* bond from $X = (x, t)$ to $Y = (y, t)$ which is open with probability $\varepsilon\lambda J_{x,y}$, and the *vertical* bond from $X = (x, t)$ to $Z = (x, t + \varepsilon)$ which is open with probability

$1-\varepsilon$. (When the ghost field is present, one also colors sites green with probability εh; the collection of all such sites is denoted $G_{L,\varepsilon}$.) All of the bond (and color, if $h>0$) random variables are independent. The measure for this process is denoted $P^{\lambda}_{L,\varepsilon}$ (or $P^{\lambda,h}_{L,\varepsilon}$ if there is a ghost field).

The important observables in the discrete model are direct analogues of their continuous time cousins (here we freely borrow the connection notation of Section 2.1):

$$M_{L,\varepsilon}(\lambda, h) = P^{\lambda,h}_{L,\varepsilon}(\mathbf{O} \to G_{L,\varepsilon}) \tag{2.25}$$

$$\chi_{L,\varepsilon}(\lambda, h) = \varepsilon \sum_{Y \in \Lambda_{L,\varepsilon}} P^{\lambda,h}_{L,\varepsilon}(\mathbf{O} \to Y, \mathbf{O} \not\to G_{L,\varepsilon}) \tag{2.26}$$

$$\chi_{L,\varepsilon}(\lambda) = \varepsilon \sum_{Y \in \Lambda_{L,\varepsilon}} P^{\lambda}_{L,\varepsilon}(\mathbf{O} \to Y) = \varepsilon \sum_{y \in \Lambda_L} \sum_{k=0}^{[L/\varepsilon]} P^{\lambda}_{L,\varepsilon}(\mathbf{O} \to (y, k\varepsilon)) \tag{2.27}$$

(where $[a]$ is the greatest integer no larger than a), and

$$\bar{\chi}_{L,\varepsilon}(\lambda) = \varepsilon \max_{z \in \Lambda_L} \sum_{Y \in \Lambda_{L,\varepsilon}} P^{\lambda}_{L,\varepsilon}((z,0) \to Y) \tag{2.28}$$

So as to avoid problems taking the limits of suprema, we take as our discretized diagrams

$$\nabla_{L,\varepsilon}(\lambda; r_1, r_2, s) = \max_{z : r_1 \leqslant |z| \leqslant r_2} \varepsilon^2 \sum_{U, V \in \Lambda_{L,\varepsilon}} P^{\lambda}_{L,\varepsilon}(\mathbf{O} \to V)\, P^{\lambda}_{L,\varepsilon}((z,s) \to U)\, P^{\lambda}_{L,\varepsilon}(U \to V) \tag{2.29}$$

$$B_{L,\varepsilon}(\lambda; r_1, r_2, s) = \max_{z : r_1 \leqslant |z| \leqslant r_2} \varepsilon \sum_{U \in \Lambda_{L,\varepsilon}} P^{\lambda}_{L,\varepsilon}(\mathbf{O} \to U)\, P^{\lambda}_{L,\varepsilon}((z,s) \to U) \tag{2.30}$$

and

$$A_{L,\varepsilon}(\lambda; r, s) = \max_{z : |z| = r} P^{\lambda}_{L,\varepsilon}(\mathbf{O} \to (z,s)) \tag{2.31}$$

It is the case that $P^{\lambda}_{L,\varepsilon} \Rightarrow P^{\lambda}_{L}$ and $P^{\lambda,h}_{L,\varepsilon} \Rightarrow P^{\lambda,h}_{L}$ (see, e.g., [BG2]); from this we can conclude that the discretized quantities in (2.25)–(2.28) converge to their continuous-time (finite-volume) counterparts, and the limits of the discretized diagrams in (2.29)–(2.31) are bounded from above by $\nabla_L(\lambda; r_1)$, $B_L(\lambda; r_1)$ and $A_L(\lambda; r)$, resp. As it may not be quite so clear how one computes the $\varepsilon \searrow 0$ limits of the quantities in (2.26)–(2.30), we demonstrate the convergence of $\chi_{L,\varepsilon}$ to χ_L—the other limits can be treated similarly. From the more detailed expression for $\chi_{L,\varepsilon}$ in (2.27), we see that it suffices to show that for each $y \in \hat{\Lambda}_L$,

$$\sum_{k=0}^{[L/\varepsilon]} \varepsilon P^{\lambda}_{L,\varepsilon}(\mathbf{O} \to (y, k\varepsilon)) \xrightarrow[\varepsilon \searrow 0]{} \int_0^L P^{\lambda}_L(\mathbf{O} \to (y, t))\, dt \tag{2.32}$$

To prove (2.32), interpret the sum on its left-hand side as the integral of a step function, and then apply the Bounded Convergence Theorem—the main reason for working in the finite volume Λ_L is being able to apply this convergence theorem when taking continuum limits such as in (2.32). See Lemma 2 in [W] for a similar, but more detailed, argument.

3. IMPLICATIONS OF THE TRIANGLE CONDITION

Propositions 3.1 and 3.2 contain the statements of the differential inequalities that are derived in Section 5, and used here to prove Theorem 1.

Proposition 3.1. Let $0 < \lambda_0 < \lambda_c$ and assume that the triangle condition (2.13) is satisfied. Then there exists a (positive) constant c such that for every $0 < h_0 < h_1 < \infty$ we can find an L_0 for which

$$\frac{\partial (M_L)^2}{\partial h} \leqslant c^2 \tag{3.1}$$

on all of $[\lambda_0, \lambda_c] \times [h_0, h_1]$ whenever $L \geqslant L_0$.

Proposition 3.2. (a) For all $\lambda > 0$ and L,

$$\frac{d\bar{\chi}_L}{d\lambda} \leqslant |J|\, (\bar{\chi}_L)^2 \tag{3.2}$$

(b) Assume that the triangle condition (2.13) is satisfied. Then there exists a λ_0 in $(0, \lambda_c)$ and a (positive) constant c such that for every $\lambda_1 \in (\lambda_0, \lambda_c)$ there is an L_0 for which

$$\frac{d\chi_L}{d\lambda} \geqslant \frac{1}{c} (\chi_L)^2 \tag{3.3}$$

on $[\lambda_0, \lambda_1]$ whenever $L \geqslant L_0$.

Proof of Theorem 1a. We only need to prove the upper bound on $M(\lambda_c, h)$ in (2.17), as the other bound was already proven in [BG2]. Integrate (3.1) from h_0 to h_1 with λ held fixed at some $\lambda' \in [\lambda_0, \lambda_c)$, next take the infinite-volume limit, and then send h_0 to zero to obtain

$$M(\lambda', h_1)^2 \leqslant c^2 h_1 \tag{3.4}$$

(Recall from (2.10) and (2.2) that $\lim_{h_0 \searrow 0} M_L(\lambda', h_0) = \theta(\lambda') = 0$.) Recall that

$$M(\lambda, h) = 1 - E^{\lambda}(e^{-h \|C\|}) = 1 - \int_0^{\infty} e^{-ht} \, dP^{\lambda}(\|C\| < t) \tag{3.5}$$

By a standard argument, $P^{\lambda}(\|C\| < t)$ is a continuous function of λ for any $t > 0$ (it only depends on the process up to a finite time). So $M(\lambda, h)$ is a continuous function of λ for any $h > 0$ by the Helly-Bray Theorem (cf. [CT]). As (3.4) is a uniform bound for all $\lambda' \in [\lambda_0, \lambda_c)$, we get the desired upper bound on $M(\lambda_c, h)$ by letting $\lambda' \nearrow \lambda_c$. ∎

Proof of Theorem 1b. The lower bound in (2.15) was proven in [BG2]. The upper bound follows from the upper bound in (2.17) by a set of extrapolation principles proven in [AF] (also see [N])—where they were used for the Ising model—and restated in [AB] and [BA]—where they were applied to percolation. These extrapolation principles use a Burgers inequality (proven for the contact process in [BG2]) to relate critical behavior along different rays through the critical point ($\lambda = \lambda_c$, $h = 0$). In this particular context, they tell us that the upper bound in (2.17) implies the upper bound in (2.15). ∎

Proof of Theorem 1c. By integrating (3.2) from λ' ($< \lambda_c$) to λ'' ($> \lambda_c$), we get

$$[\bar{\chi}_L(\lambda')]^{-1} - [\bar{\chi}_L(\lambda'')]^{-1} \leqslant |J| \, (\lambda'' - \lambda') \tag{3.6}$$

It follows from (2.4) and (2.24) that $\lim_{L \to \infty} (\bar{\chi}_L)^{-1}$ vanishes at $\lambda = \lambda''$, and so (using (2.24) again) we have

$$[\chi(\lambda')]^{-1} \leqslant |J| \, (\lambda'' - \lambda') \tag{3.7}$$

Now let $\lambda'' \searrow \lambda_c$ to obtain the lower bound on χ in (2.16).

Under the triangle condition, we also have inequality (3.3) which we integrate over $[\lambda, \lambda_1]$ (with $\lambda_1 < \lambda_c$). One then takes the infinite-volume limit, and sends λ_1 to λ_c to get the desired upper bound on χ. ∎

4. DELOCALIZATION AND FACTORIZATION

Throughout this section, we use P to represent the discretized finite-volume probability measure, $P^{\lambda, h}_{L, \varepsilon}$.

4.1. Delocalization

We will show how to bound the probability of an event $\mathscr{E}(\tilde{X}, \tilde{X}'; D; \tilde{n})$ (defined immediately below) from below in terms of the probability of a *point-split* version of that event $\mathscr{E}_s(X, X'; D; n; r)$ (defined further below). Fix a pair of horizontally (i.e., spatially) neighboring sites, $\tilde{X}$ and $\tilde{X}'$, in $\Lambda_{L, \varepsilon}$, let $\tilde{n}$ be either 0 or 1, and take D to be either $G_{L, \varepsilon}$ (the ghost set) or else some deterministic subset of $\Lambda_{L, \varepsilon}$. Then $\mathscr{E}(\tilde{X}, \tilde{X}'; D; \tilde{n})$ is the event that

- $\mathbf{O} \to \tilde{X}$, but $\mathbf{O} \not\to D$ off $\{\tilde{X}\}$, and
- $\tilde{X}'$ is connected to some site in D, $\tilde{X}$ is connected to $\tilde{n}$ site(s) in D, and if $\tilde{n} = 1$ the two connections to D may be traced in site-disjoint way.

Of course, if $\tilde{n} = 0$, the condition that $\tilde{X}$ be connected to 0 sites in D is vacuously satisfied.

To prove Proposition 3.1, we set $D = G_{L, \varepsilon}$ and $\tilde{n} = 1$, while for Proposition 3.2, D is a site $Y \in \Lambda_{L, \varepsilon}$ (over which we sum) and $\tilde{n} = 0$. Readers can make this subsection more understandable by focusing on the case $\tilde{n} = 1$—as we do. We comment that our argument is easily extended to the cases where $\tilde{n} \geqslant 2$ and where there are several disjoint connections from $\tilde{X}'$ to D—but such generality is not needed here.

There are $\tilde{n} + 2$ different connection events in the original event $\mathscr{E}$, each of which involves either $\tilde{X}$ or its neighbor $\tilde{X}'$: there is a connection arriving at $\tilde{X}$ from the origin, $\tilde{X}'$ is connected to a site in D, and $\tilde{X}$ is connected to $\tilde{n}$ sites in D. Since the correction terms which lead to the complementary inequalities arise out of intersections between these connections we want to physically separate the $\tilde{n} + 2$ connections. This will be done by having the $n = \tilde{n} + 1$ connections to D emanate not directly from $\tilde{X}$ and $\tilde{X}'$, but rather from some n *jump-off* sites moved into the future, and also spatially separated from one another.

To delocalize $\mathscr{E}$, we "rewire" the connections from $\tilde{X}$ and $\tilde{X}'$ $(= \tilde{X} + (e, 0))$ to get disjoint connections from nearby, slightly earlier sites X and X' to much later, widely separated sites Z_1 and Z_2, and from there to D; note that the Markov property of the contact process ensures that changing connections between the time of X (or X') and the time of Z_i cannot destroy connection events from $\mathbf{O}$ to X, or from Z_i to D. Specifically, we take these sites to be

$$X = \tilde{X} - \left(0, \left[\frac{1}{\varepsilon}\right] \varepsilon\right), \qquad X' = \tilde{X}' - \left(0, \left[\frac{1}{\varepsilon}\right] \varepsilon\right)$$

and

$$Z_i = X + ((-1)^{i+1} re, t) \tag{4.1}$$

where r is a positive integer, e is the spatial part of $\tilde{X}' - \tilde{X}$ (or of $X' - X$), and t is a sufficiently large positive integer multiple of ε—we shall take $t = (r+1)[1/\varepsilon]\,\varepsilon$. In Section 5 we will control the diagram functions by taking r to be sufficiently large; t is a period of time long enough for us to perform a delocalization that is well-behaved in the continuum ($\varepsilon \searrow 0$) limit.

The delocalized or (point-split) event $\mathscr{E}_s(X, X'; D; \tilde{n}+1; r)$ is defined for X, X', D, $\tilde{n}$ and r as above is the event (a special instance is illustrated in Fig. 1) that

- $\mathbf{O} \to X$, but $\mathbf{O} \not\to D$, and
- $Z_1 \to D$ off $C(\mathbf{O})$ (and, if $\tilde{n} = 1$, $Z_2 \to D$ off $C(\mathbf{O}) \cup C(Z_1)$).

In subsequent work with $\mathscr{E}_s$, it will be convenient to use $n = \tilde{n} + 1$, the required number of disjoint connections from X and X' to D, in place of $\tilde{n}$, the required number of connections from $\tilde{X}$ to D.

To compare the probabilities of $\mathscr{E}$ and $\mathscr{E}_s$ we will introduce in the proof of Theorem 4.1 a pair of (independent) events $\mathscr{F}$ and $\mathscr{G}$ with the properties that $\mathscr{E}_s$ is a subevent of $\mathscr{F}$, and $\mathscr{F} \cap G$ is contained in $\mathscr{E}$. For such a pair of events, we have

$$P(\mathscr{E}_s)\, P(\mathscr{G}) \leqslant P(\mathscr{F})\, P(\mathscr{G}) = P(\mathscr{F} \cap G) \leqslant P(\mathscr{E}) \tag{4.2}$$

Roughly speaking, the independence of $\mathscr{F}$ and $\mathscr{G}$ will follow from the fact that one event belongs to a cylinder over some subset B of $\Lambda_{L,\varepsilon}$, and the other belongs to the cylinder over the complement of B. The set B has the form of a $1+1$ dimensional space-time box in $\Lambda_{L,\varepsilon}$ (see Fig. 1):

$$B = B(X; r, \tau) = \{X + (ke, s) : -r \leqslant k \leqslant r, 0 \leqslant s \leqslant \tau\} \tag{4.3}$$

Observe that the sites Z_1 and Z_2 are on the top of $B(X; r, t)$.

Theorem 4.1 (Delocalization). Let $\mathscr{E} = \mathscr{E}(\tilde{X}, \tilde{X}'; D; \tilde{n})$ and $\mathscr{E}_s = \mathscr{E}_s(X, X'; D; n; r)$ be as defined above, with $\tilde{X}, \tilde{X}', X, X', Z_1, Z_2 \in \Lambda_{L,\varepsilon}$, $n = \tilde{n} + 1$, and either $D = G_{L,\varepsilon}$ or else D is a deterministic set that is not too

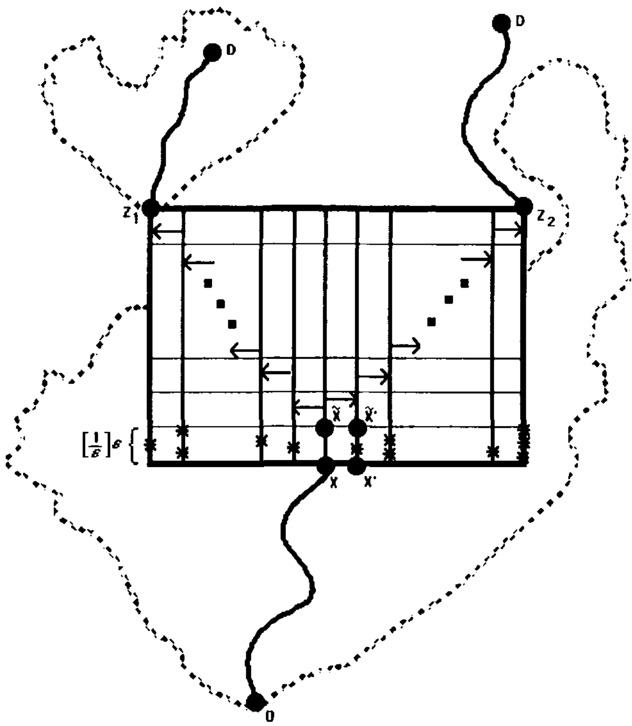

Fig. 1. The intersection of the events F and G. Outside of the box $B(X; r, t)$, bold lines indicate connections made along open bonds, and the dashed lines indicate the surfaces surrounding various clusters. Inside $B(X; r, t)$, the arrows indicate open bonds, and the asterisks mark deaths (closed vertical bonds).

close to X in the sense that $D \cap B(X; r, [1/\varepsilon]\, \varepsilon) = \emptyset$. Let $0 < \lambda_0 \leqslant \lambda_1 < \infty$ and $0 \leqslant h_1 < \infty$. Then there exists a positive constant $a = a(\lambda_0, \lambda_1, h_1; J_{0,e}; r, t)$ such that, for $(\lambda, h) \in [\lambda_0, \lambda_1] \times [0, h_1]$,

$$P(\mathscr{E}) \geqslant a P(\mathscr{E}_s) \tag{4.4}$$

Proof. Our plan is to define events $\mathscr{F}$ and $\mathscr{G}$ for which we are able to verify (4.2), and then to calculate $P(\mathscr{G})$. We begin by considering the event $\mathscr{F} = \mathscr{F}(X, X'; D; n; r)$ (see Fig. 1) that

- $\mathbf{O} \to X$, but $\mathbf{O} \not\to D$ off $B(X; r, t)$, and
- $Z_1 \to D$ off $C(\mathbf{O})$ (and, if $n = 2$, $Z_2 \to D$ off $C(\mathbf{O}) \cup C(Z_1)$).

It is clear that $\mathscr{E}_s(X, X'; D; n; r) \subset \mathscr{F}$, which gives us the first inequality in (4.2).

We next need to find a "rewiring" event $\mathscr{G}$ in the cylinder over $B(X; r, t)$ whose intersection with $\mathscr{F}$ implies the occurence of $\mathscr{E}$. To define such an event, we need to (i) make certain that there are no deaths along the common time-line of X and $\tilde{X}$ in the first macroscopic time unit (so that $\mathbf{O} \to \tilde{X}$), (ii) seal off the bottom of the box B with deaths (except along the time-line containing X) and keep any horizontal bonds from entering or leaving B (so that the origin is not connected to D), and (iii) build connections inside B from $\tilde{X}$ and $\tilde{X}'$ to the appropriate Z_i's—say, along a (pair of) *staircase*(s) of horizontal bonds. So we now define $\mathscr{G} = \mathscr{G}(X, X'; D; r)$ to be the event (see Fig. 1) that

(1) no sites from D are within $[1/\varepsilon]\,\varepsilon$ time units of the bottom of the box: $D \cap B(X; r, [1/\varepsilon]\,\varepsilon) = \varnothing$,

(2) each time-line, except the one containing X, has at least one death in the box within $[1/\varepsilon]\,\varepsilon$ time units of the bottom of the box,

(3) there are no other deaths in $B(X; r, t)$ besides the ones required in (2),

(4) for each $k = 2,..., r$, there exist "times" $s_{k,1}$ and $s_{k,2}$ with $k[1/\varepsilon]\,\varepsilon < s_{k,1}, s_{k,2} \leqslant k[1/\varepsilon]\,\varepsilon$ such that there are open horizontal bonds from $X + ((k-1)\,e, s_{k,1})$ to $X + (ke, s_{k,1})$ and from $X + ((1-k)\,e, s_{k,2})$ to $X + (-ke, s_{k,2})$ (—the possibility of there being several times $s_{k,i}$ is allowed), for $k = 1$ there exists a time (or times) $s_{1,2}$, with $[1/\varepsilon]\,\varepsilon < s_{1,2} \leqslant 2[1/\varepsilon]\,\varepsilon$, at which there is an open horizontal bond from $X + (0, s_{1,2})$ to $X + (-e, s_{1,2})$, and

(5) there are no other open horizontal bonds having one or both endpoints in $B(X; r, t)$ aside from those mentioned in (4).

Notice that $\mathscr{G}$ and $\mathscr{F}$ depend on nonoverlapping sets of random variables; hence we have the equality in (4.2). Next complete the proof of (4.2) by observing that $\mathscr{E}$ must occur if both $\mathscr{F}$ and $\mathscr{G}$ do.

It only remains to calculate $P(\mathscr{G})$. We begin by writing

$$P(\mathscr{G}) = P(\mathscr{G}_1) \cdot P(\mathscr{G}_2) \cdot P(\mathscr{G}_3) \cdot P(\mathscr{G}_4) \cdot P(\mathscr{G}_5) \tag{4.5}$$

where $\mathscr{G}_1$, $\mathscr{G}_2$, $\mathscr{G}_3$, $\mathscr{G}_4$, and $\mathscr{G}_5$ are the five (independent) events listed in the definition of $\mathscr{G}$: for example, $\mathscr{G}_1 = \{D \cap B(X; r, [1/\varepsilon]\,\varepsilon) = \varnothing\}$. In the case where D is a deterministic set, then (by the hypotheses of the the theorem) $P(\mathscr{G}_1) = 1$. If $D = G_{L,\varepsilon}$, then $P(\mathscr{G}_1) = (1 - \varepsilon h)^{([1/\varepsilon]+1)(2r+1)}$. In either case

$$P(\mathscr{G}_1) \geqslant \exp[-h(2r+1)] + o(1) \tag{4.6}$$

as $\varepsilon \mathbf{z} 0$. Even more simply,

$$P(\mathscr{G}_2) = (1 - \exp[-1])^{2r} + o(1) \quad \text{and} \quad P(\mathscr{G}_3) = \exp[-2r^2 - (r+1)] + o(1) \tag{4.7}$$

We also have the crude bounds

$$P(\mathscr{G}_4) \geqslant (1 - \exp[-\lambda J_{0,e}])^{2r-1} + o(1) \tag{4.8}$$

and

$$P(\mathscr{G}_5) \geqslant \exp[-\lambda \, |J| \, (r+1)(2r+1)] \tag{4.9}$$

Using the various bounds and approximations (4.6)–(4.9) in (4.5), we get

$$P(\mathscr{G}) \geqslant f(\lambda, h) + o(1) \tag{4.10}$$

where f is continuous and positive on $[\lambda_0, \lambda_1] \times [0, h_1]$. The desired relation (4.4) between $P(\mathscr{E})$ and $P(\mathscr{E}_s)$ now follows (for ε sufficiently small) from (4.2) and (4.10). ▮

4.2. Factorization

Let us first introduce some necessary notation: a subset A of $\Lambda_{L,\varepsilon}$ has a horizontal boundary

$$\partial_H(A) = \{U \in A : U \text{ is a horizontal neighbor of a site } U' \notin A\} \tag{4.11}$$

and a vertical boundary

$$\partial_V(A) = \{U \in A : U'' = U - (0, \varepsilon) \notin A\} \tag{4.12}$$

Let $I[\cdot]$ be the indicator function which is 1 if the condition in the brackets holds, and 0 otherwise. If $U' = (u', s)$ and $U = (u, t)$, we agree to write $J_{U', U} = J_{u', u} I[s = t]$.

We next state some diagrammatic inequalities whose proofs (see [AN], [D1] and [BA]) can be based on the van den Berg-Kesten-Fiebig (BKF) and Hammersley-Simon-Lieb (HSL) inequalities. We use a detailed HSL inequality (similar to that of [CKP]) which distinguishes between contributions from the horizontal and vertical boundaries, so as to be able to extract explicit factors of ε from connections made across horizontal bonds—these factors are necessary in order to see that the finite-volume expressions scale correctly as $\varepsilon \searrow 0$.

Proposition 4.2. For any $A, D \subset \Lambda_{L,\varepsilon}$ and $X, Y, Z \in \Lambda_{L,\varepsilon}$,

$$\begin{aligned} &P(X \to D) \\ &\quad \geqslant P(X \to D \text{ off } A) \\ &\quad \geqslant P(X \to D) - \sum_{U',\, U:\, U \in \partial_H(A)} \varepsilon\lambda J_{U',\,U} P(X \to U')\, P(U \to D) \\ &\qquad - \sum_{U \in \partial_V(A)} P(X \to U)\, P(U \to D) - I[X \in A]\, P(X \to D) \end{aligned} \tag{4.13}$$

$$\begin{aligned} &P(X \to Y, X \to D) \\ &\quad \leqslant \sum_{V',\, V} \varepsilon\lambda J_{V',\,V} P(X \to V')\, P(V' \to Y)\, P(V \to D) \\ &\qquad + \sum_{V',\, V} \varepsilon\lambda J_{V',\,V} P(X \to V')\, P(V' \to D)\, P(V \to Y) \\ &\qquad + P(X \to Y)\, P(Y \to D) + \sum_{V} P(X \to V, V \in D)\, P(V \to Y) \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} &P(X \to Y, X \to Z, X \nrightarrow D) \\ &\quad \leqslant \sum_{V',\, V} \varepsilon\lambda J_{V',\,V} P(X \to V', X \nrightarrow D)\, P(V' \to Y)\, P(V \to Z) \\ &\qquad + \sum_{V',\, V} \varepsilon\lambda J_{V',\,V} P(X \to V', X \nrightarrow D)\, P(V' \to Z)\, P(V \to Y) \\ &\qquad + P(X \to Y, X \nrightarrow D)\, P(Y \to Z) \\ &\qquad + P(X \to Z, X \nrightarrow D)\, P(Z \to Y) \end{aligned} \tag{4.15}$$

We are now ready to factorize the probabilities of the point-split events introduced in the preceding subsection. Although the formulas appearing in the statement of Theorem 4.3 (the factorization result) may appear rather forbidding, the proof (see the appendix) is reasonably straightforward, and simply consists of repeated applications of the diagrammatic inequalities from Proposition 4.2. In advance of the statement of this result, we make several remarks that we hope will add to the reader's understanding.

Remarks. 1. The upper bound on the probability of $\mathscr{E}_s$ is the first term in an inclusion-exclusion expansion, and the lower bound also contains the second order (or correction) terms. The reader may find it helpful if we explain our notation for labeling the correction terms $Q_{*,i,\ell}$. All of these terms have their basis in applications of (4.13) with Z_i playing the role of X and the cluster of either Z_1 (if $\ell=1$) or the origin (when $\ell=0$) being the set A. We will obtain Q_{B1}, Q_{B2} and Q_{B3} as recombinations of horizontal and vertical **B**oundary terms, Q_V is a special term which arises in considering the **V**ertical boundary terms, and Q_{I1}, Q_{I2} and Q_{I3} can be traced to the **I**ndicator function in (4.13).

2. The proof is written primarily for when $n=1$. The reader who is particularly interested in the case $n=2$ might also find it useful to consult Section 3.2 in [BA].

3. This theorem can be readily generalized to the case of any natural number n, with the major difference being that the $Q_{*,i,\ell}$'s pick up some additional factors, e.g. $Q_{B1,2,1}$ should be multiplied by $\prod_{3\leqslant k\leqslant n} P(Z_k\to D)$ when $n>2$ and $\ell\neq 0$.

4. The factor $P(Z_{-1}\to D)$ appearing in $Q_{B1,1,0}$ and $Q_{B2,1,0}$ when $n=1$ should be regarded as being identically 1. The only significance of n^2-i-1 in (4.17) and (4.18) is that it is a quantity which assumes the values 1 (resp. 2, resp. neither 1 nor 2) when $n=i=2$ (resp. $n=2$ and $i=1$, resp. $n=i=1$).

Theorem 4.3 (Factorization). Let $\mathscr{E}_s(X, X'; D; n; r)$ be as above. Then

$$
\begin{aligned}
P(0\to X, \mathbf{O}\nrightarrow D) &\prod_{i=1}^{n} P(Z_i\to D)\\
&\geqslant P(\mathscr{E}_s(X, X'; D; n; r))\\
&\geqslant P(\mathbf{O}\to X, \mathbf{O}\nrightarrow D)\prod_{i=1}^{n} P(Z_i\to D)\\
&\quad-\sum_{\substack{1\leqslant i\leqslant n\\ 0\leqslant \ell<i}}[Q_{B1,i,\ell}+Q_{B2,i,\ell}]-\sum_{\substack{2\leqslant i\leqslant n\\ 1\leqslant \ell<i}}[Q_{B3,i,\ell}+Q_{V,i,\ell}]\\
&\quad-\sum_{1\leqslant i\leqslant n}[Q_{I1,i,0}+Q_{I2,i,0}]-\sum_{\substack{2\leqslant i\leqslant n\\ 1\leqslant \ell<i}} Q_{I3,i,\ell}
\end{aligned}
\tag{4.16}
$$

where the general boundary correction terms are

$$Q_{B1,i,\ell}=\begin{cases}(\varepsilon\lambda)^2 P(\mathbf{O}\to X, \mathbf{O}\nrightarrow D) \displaystyle\sum_{U',U,V',V} J_{U',U}J_{V',V}P(Z_1\to V') \\ \quad \cdot [P(V'\to U)\,P(V\to D)+P(V'\to D)\,P(V\to U)] \\ \quad \cdot P(Z_2\to U')[P(U\to D)+P(U'\to D)], \quad \text{if } i=2,\ \ell=1 \\ (\varepsilon\lambda)^2 P(Z_{n^2-i-1}\to D) \displaystyle\sum_{U',U,V',V} J_{U',U}J_{V',V}P(\mathbf{O}\to V', \mathbf{O}\nrightarrow D) \\ \quad \cdot [P(V'\to X)\,P(V\to U)+P(V'\to U)\,P(V\to X)] \\ \quad \cdot P(Z_i\to U')[P(U\to D)+P(U'\to D)], \quad \text{if } \ell=0\end{cases} \tag{4.17}$$

$$Q_{B2,i,\ell}=\begin{cases}\varepsilon\lambda P(\mathbf{O}\to X, \mathbf{O}\nrightarrow D) \displaystyle\sum_{U',U} J_{U',U}P(Z_1\to U)\,P(U\to D) \\ \quad \cdot P(Z_2\to U')[P(U\to D)+P(U'\to D)], \quad \text{if } i=2,\ \ell=1 \\ \varepsilon\lambda P(Z_{n^2-i-1}\to D) \displaystyle\sum_{U',U} J_{U',U}P(\mathbf{O}\to X, \mathbf{O}\nrightarrow D) \\ \quad \cdot P(X\to U)\,P(Z_i\to U')[P(U\to D)+P(U'\to D)], \quad \text{if } \ell=0\end{cases} \tag{4.18}$$

$$Q_{B3,2,1}=\varepsilon\lambda P(\mathbf{O}\to X, \mathbf{O}\nrightarrow D) \sum_{U',U,V} J_{U',U}P(Z_1\to V,\ V\in D) \cdot P(V\to U)\,P(Z_2\to U')[P(U\to D)+P(U'\to D)] \tag{4.19}$$

the special vertical boundary correction term is

$$Q_{V,2,1}=P(\mathbf{O}\to X, \mathbf{O}\nrightarrow D)\,P(Z_1\to D)^2\,P(Z_2\to Z_1) \tag{4.20}$$

and the indicator function correction terms are

$$Q_{I1,i,0}=\varepsilon\lambda \prod_{k=1}^{n} P(Z_k\to D) \sum_{V',V} J_{V',V}P(\mathbf{O}\to V', \mathbf{O}\nrightarrow D) \cdot [P(V'\to X)\,P(V\to Z_i)+P(V'\to Z_i)\,P(V\to X)], \quad \text{for } i=1,2 \tag{4.21}$$

$$Q_{I2,i,0}=P(\mathbf{O}\to X, \mathbf{O}\nrightarrow D) \prod_{k=1}^{n} P(Z_k\to D)\,P(X\to Z_i), \quad \text{for } i=1,2 \tag{4.22}$$

and

$$Q_{I3,2,1} = P(\mathbf{O} \to X, \mathbf{O} \nrightarrow D)\, P(Z_2 \to D)\, P(Z_1 \to Z_2, Z_1 \to D) \qquad (4.23)$$

5. DIFFERENTIAL INEQUALITIES

5.1. An Inequality for $M(\lambda, h)$

Proof of Proposition 3.1. We begin as in [BA], by observing that it is possible to relate $M_{L,\varepsilon}$ to a sum of the probabilities of the events $\mathscr{E}$ introduced in Section 4. Specifically, one way for the event "$\mathbf{O} \to G_{L,\varepsilon}$" to occur is if there are at least two green sites in the cluster of the origin, yet the origin is not doubly connected to $G_{L,\varepsilon}$. In such a situation, there must be (see [AB], Lemma 3.5) a unique site $\tilde{X}$ such that "$\mathbf{O} \to \tilde{X}$," "$\mathbf{O} \nrightarrow G_{L,\varepsilon}$ off $\{\tilde{X}\}$" and "$\tilde{X}$ is doubly connected to $G_{L,\varepsilon}$." With an additional loss of probability, one could assume that the double connection from $\tilde{X}$ happens in the following way: find some $e \in \mathbb{Z}^d$ for which $J_{0,e} > 0$, and then suppose that the horizontal bond from $\tilde{X}$ to its neighbor $\tilde{X}' = \tilde{X} + (e, 0)$ is open and that there are site-disjoint paths (of open horizontal and vertical bonds) connecting $\tilde{X}$ and $\tilde{X}'$ to green sites. For technical reasons, we choose to restrict $\tilde{X}$ to $(0, [1/\varepsilon]\,\varepsilon) + \Lambda_{L/3,\varepsilon}$. In the notation of Section 4, we have that

$$M_{L,\varepsilon} = P^{\lambda,h}_{L,\varepsilon}(\mathbf{O} \to G_{L,\varepsilon}) \geqslant \varepsilon\lambda J_{0,e} \sum_{\tilde{X} \in (0,[1/\varepsilon]\varepsilon) + \Lambda_{L/3,\varepsilon}} P^{\lambda,h}_{L,\varepsilon}(\mathscr{E}(\tilde{X}, \tilde{X}'; G_{L,\varepsilon}; 1)) \qquad (5.1)$$

Apply the delocalization procedure (Theorem 4.1) with $\tilde{n} = 1$ and $D = G_{L,\varepsilon}$ to get that for $\lambda_0 \leqslant \lambda \leqslant \lambda_c$ and $0 < h \leqslant h_1$ (for some positive h_1), there exists a positive constant a such that

$$M_{L,\varepsilon} \geqslant aJ_{0,e}\,\varepsilon\lambda \sum_{X \in \Lambda_{L/3,\varepsilon}} P^{\lambda,h}_{L,\varepsilon}(\mathscr{E}_s(X, X'; G_{L,\varepsilon}; 2; r)) \qquad (5.2)$$

where r will be determined later. Next, use the factorization procedure (Theorem 4.3) to obtain an even lower bound which can be represented as a single leading (positive) term,

$$\mathscr{L} = \varepsilon \sum_{X \in \Lambda_{L/3,\varepsilon}} P^{\lambda,h}_{L,\varepsilon}(\mathbf{O} \to X, \mathbf{O} \nrightarrow G_{L,\varepsilon})\, P^{\lambda,h}_{L,\varepsilon}(Z_1 \to G_{L,\varepsilon})\, P^{\lambda,h}_{L,\varepsilon}(Z_2 \to G_{L,\varepsilon}) \qquad (5.3)$$

less 13 correction terms, which are ε times the sums over X of $Q_{B1,2,1}$, $Q_{B1,2,0}$, $Q_{B1,1,0}$, $Q_{B2,2,1}$, $Q_{B2,2,0}$, $Q_{B2,1,0}$, $Q_{B3,2,1}$, $Q_{V,2,1}$, $Q_{I1,2,0}$,

$Q_{I1,1,0}$, $Q_{I2,2,0}$, $Q_{I2,1,0}$, and $Q_{I3,2,1}$—with the entire difference multiplied by $aJ_{0,\varepsilon}\lambda$.

To find upper bounds for the sums of the correction terms, we use the facts that

$$P^{\lambda,h}_{L,\varepsilon}(Z \to G_{L,\varepsilon}) \leqslant P^{\lambda,h}_{2L,\varepsilon}(\mathbf{O} \to G_{2L,\varepsilon}) = M_{2L,\varepsilon} \tag{5.4}$$

for any site $Z \in \Lambda_{L,\varepsilon}$ (the "volume-doubling" compensates for the absence of translation invariance due to the free boundary conditions), and also

$$\varepsilon \sum_{Z} P^{\lambda,h}_{L,\varepsilon}(\mathbf{O} \to Z, \mathbf{O} \nrightarrow G_{L,\varepsilon}) \leqslant \chi_{L,\varepsilon} \tag{5.5}$$

(Note that we can relax the restriction of X to $\Lambda_{L/3,\varepsilon}$ when we use (5.5) to get upper bounds on the corrections.) Straightforward applications of (5.4) and (5.5) yield the estimates

$$\varepsilon \sum_{X} Q_{B1,i,0} \leqslant 4(\lambda\,|J|)^2\,\nabla_{2L,\varepsilon}(\lambda; r-2R, r+2R, t)(M_{2L,\varepsilon})^2\,\chi_{L,\varepsilon}, \quad \text{for } i=1,2 \tag{5.6}$$

$$\varepsilon \sum_{X} Q_{B1,2,1} \leqslant 4(\lambda\,|J|)^2\,\nabla_{2L,\varepsilon}(\lambda; 2r-2R, 2r+2R, 0)(M_{2L,\varepsilon})^2\,\chi_{L,\varepsilon} \tag{5.7}$$

$$\varepsilon \sum_{X} Q_{B2,i,0},\ \varepsilon \sum_{X} Q_{I1,i,0} \leqslant 2\lambda\,|J|\,B_{2L,\varepsilon}(\lambda; r-R, r+R, t)(M_{2L,\varepsilon})^2\,\chi_{L,\varepsilon}, \quad \text{for } i=1,2 \tag{5.8}$$

$$\varepsilon \sum_{X} Q_{B2,2,1} \leqslant 2\lambda\,|J|\,B_{2L,\varepsilon}(\lambda; 2r-R, 2r+R, 0)(M_{2L,\varepsilon})^2\,\chi_{L,\varepsilon} \tag{5.9}$$

$$\varepsilon \sum_{X} Q_{V,2,1} \leqslant A_{2L,\varepsilon}(\lambda; 2r, 0)(M_{2L,\varepsilon})^2\,\chi_{L,\varepsilon} \tag{5.10}$$

and

$$\varepsilon \sum_{X} Q_{I2,i,0} \leqslant A_{2L,\varepsilon}(\lambda; r, t)(M_{2L,\varepsilon})^2\,\chi_{L,\varepsilon}, \qquad \text{for} \quad i=1,2 \tag{5.11}$$

To aid the reader in filling in the steps that lead to (5.6)–(5.11), we sketch the derivation of (5.6) for the case when $i=1$. Applying (5.4) twice, one has

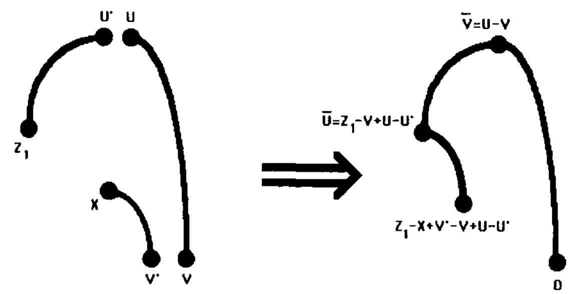

Fig. 2. Translation of connection events to build the triangle in (5.12). Each edge indicates the probability of a connection event.

that $\varepsilon \sum_X Q_{B1,1,0}$ is bounded above by the sum of two similar terms, one of which is

$$2\varepsilon^3\lambda^2(M_{2L,\varepsilon})^2 \sum_{V'} P^{\lambda,h}_{L,\varepsilon}(\mathbf{O} \to V', \mathbf{O} \not\to G_{L,\varepsilon})$$
$$\times \sum_{\substack{X,V \\ U,U'}} J_{U',U} J_{V',V} P^{\lambda,h}_{L,\varepsilon}(V' \to X)\, P^{\lambda,h}_{L,\varepsilon}(V \to U)\, P^{\lambda,h}_{L,\varepsilon}(Z_1 \to U') \tag{5.12}$$

Translate the last three connection events (paying the volume-doubling price) as indicated in Figure 2, and introduce new summation variables $\bar{V} = U - V$ and $\bar{U} = Z_1 - V + U - U'$. Summing first over X, next over U, then over $\bar{V}$ and $\bar{U}$, and finally over V' gives us the two factors of $|J|$, a triangle "diagram," and $\chi_{L,\varepsilon}$. Note that this triangle has been opened by the spatial difference between $Z_1 - X + V' - V + U - U'$ and the origin, that the spatial distance between Z_1 and X must be at least r (by the delocalization procedure), and that the spatial differences $V' - V$ and $U - U'$ can each be no larger than the maximum range R. In a similar fashion, one can establish the rest of (5.6)–(5.11).

Each of the remaining two corrections contains an unusual term requiring special attention. First, in $Q_{B3,2,1}$, it is not difficult to see that

$$P(Z_1 \to V, V \in G_{L,\varepsilon}) = \varepsilon h\, P(Z_1 \to V) \tag{5.13}$$

so

$$\varepsilon \sum_X Q_{B3,2,1} \leqslant 2\lambda\, |J|\, \nabla_{2L,\varepsilon}(\lambda; 2r - 2R, 2r + 2R, 0) \left(\frac{h}{M_{2L,\varepsilon}}\right) (M_{2L,\varepsilon})^2 \chi_{L,\varepsilon} \tag{5.14}$$

Finally, an analysis of the term $P_{L,\varepsilon}^{\lambda,h}(Z_1 \to Z_2, Z_1 \to G_{L,\varepsilon})$—recall that Z_1 and Z_2 have the same time coordinate but are widely separated in space—shows that, as $\varepsilon \searrow 0$,

$$\varepsilon\lambda \sum_X Q_{I3,2,1} = o(1)(M_{2L,\varepsilon})^2 \chi_{L,\varepsilon} \tag{5.15}$$

Turning to the leading term, $\mathscr{L}$, we get a lower bound on $P_{L,\varepsilon}^{\lambda,h}(Z_i \to G_{L,\varepsilon})$ by requiring the connection to take place inside $Z_i + \Lambda_{L/3,\varepsilon}$:

$$P_{L,\varepsilon}^{\lambda,h}(Z_i \to G_{L,\varepsilon}) \leqslant M_{L/3,\varepsilon} \tag{5.16}$$

Using the bounds (5.6)–(5.16) in what we got by factorizing (5.2), and taking the $\varepsilon \searrow 0$ limit, we obtain

$$\begin{aligned} M_L(\lambda, h) \geqslant aJ_{0,e}\lambda \Bigg[&(M_{L/3}(\lambda, h))^2 \sum_{x' \in \Lambda_{L/3}} \int_0^{L/3} P_L^{\lambda,h}(\mathbf{O} \to (x', t), \mathbf{O} \nrightarrow G_L)\, dt \\ &- 2\left(6(\lambda_c |J|)^2 + 10\lambda_c |J| + 6 + \lambda_c |J| \frac{h}{M_{2L}(\lambda, h)}\right) \\ &\times \nabla(\lambda_c, r - 2R)(M_{2L}(\lambda, h))^2 \chi_L(\lambda, h)\Bigg] \end{aligned} \tag{5.17}$$

for $\lambda_0 \leqslant \lambda \leqslant \lambda_c$, where we have also made use of (2.20) and the monotonicity properties of the diagram functions in both L and λ.

The term h/M is bounded near the critical point ($\lambda = \lambda_c$, $h = 0_+$) since the concavity of M in h tells us that $h/M < 1/\chi$. Thus for any (λ, h) with $\lambda_0 \leqslant \lambda \leqslant \lambda_c$ and $0 < h \leqslant h_1$ there is a (positive) constant K such that

$$\begin{aligned} M_L(\lambda, h) \geqslant aJ_{0,e}\lambda \Bigg[&(M_{L/3}(\lambda, h))^2 \sum_{x' \in \Lambda_{L/3}} \int_0^{L/3} P_L^{\lambda,h}(\mathbf{O} \to (x', t), \mathbf{O} \nrightarrow G_L)\, dt \\ &- K\,\nabla(\lambda_c, r - 2R)(M_{2L}(\lambda, h))^2 \chi_L(\lambda, h)\Bigg] \end{aligned} \tag{5.18}$$

Now we know that $M_{L/3}$, M_L, and M_{2L} all converge to M as $L \to \infty$ for all $\lambda \in [0, \infty)$ and $h > 0$. Furthermore, $M(\lambda, h)$ is a positive function which is increasing in both λ and h on $(0, \infty)^2$. Letting $\alpha = M(\lambda_0, h_0)/2$, we can use a compactness argument to show that for every $0 < \lambda_0 \leqslant \lambda_c \leqslant \lambda_1 < \infty$ and $0 < h_0 \leqslant h_1 < \infty$, there is an L sufficiently large so that

$$|M_{L/3}(\lambda, h) - M(\lambda, h)|,\ |M_L(\lambda, h) - M(\lambda, h)|,\ |M_{2L}(\lambda, h) - M(\lambda, h)| < \alpha \tag{5.19}$$

on $[\lambda_0, \lambda_1] \times [h_0, h_1]$. It follows from (5.19) that for large enough L,

$$M_{L/3}(\lambda, h) \geqslant \tfrac{1}{3} M_L(\lambda, h) \qquad \text{and} \qquad M_{2L}(\lambda, h) \leqslant 3M_L(\lambda, h) \tag{5.20}$$

uniformly on $[\lambda_0, \lambda_1] \times [h_0, h_1]$. By a related argument we also have that if L is sufficiently large, then for all $\lambda_0 \leqslant \lambda \leqslant \lambda_1$ and $h_0 \leqslant h \leqslant h_1$,

$$\sum_{x' \in \Lambda_{L/3}} \int_0^{L/3} dt\, P_L^{\lambda, h}(\mathbf{O} \to (x', t), \mathbf{O} \nrightarrow G_L) \geqslant \frac{1}{3} \chi_L(\lambda, h) = \frac{1}{3} \frac{\partial M_L(\lambda, h)}{\partial h} \tag{5.21}$$

Using the bounds (5.20) and (5.21) in (5.18) gives us

$$M_L \geqslant a J_{0,e} \lambda_0 \left[\frac{1}{27} - 9K \nabla(\lambda_c, r - 2R) \right] (M_L)^2 \frac{\partial M_L}{\partial h} \tag{5.22}$$

on $[\lambda_0, \lambda_c] \times [h_0, h_1]$. When the triangle condition (2.13) is satisfied, we can make the bracketed factor in (5.22) positive by taking r sufficiently large, and this proves inequality (3.1). ∎

5.2. A Pair of Inequalities for $\chi(\lambda)$

Proof of Proposition 3.2. We first briefly indicate how one obtains the inequality (3.2), which holds for all λ—here we follow [AN] (also see [D1]). Let z be a site in $\hat{\Lambda}_L$ which maximizes the sum on the right-hand side of (2.28), and denote that sum by $\chi_{L,\varepsilon}(\lambda; z)$. By Russo's formula (cf. [Gr]),

$$\begin{aligned} \frac{d\chi_{L,\varepsilon}(\lambda; z)}{d\lambda} &= \varepsilon^2 \sum_{\tilde{X}, \tilde{X}', Y \in \Lambda_{L,\varepsilon}} J_{\tilde{X}, \tilde{X}'} P(\text{the bond from } \tilde{X} \text{ to } \tilde{X}' \text{ is pivotal for } (z, 0) \to Y) \\ &= \varepsilon^2 \sum_{\tilde{X}, \tilde{X}', Y \in \Lambda_{L,\varepsilon}} \frac{J_{\tilde{X}, \tilde{X}'}}{1 - \varepsilon\lambda J_{\tilde{X}, \tilde{X}'}} P((z, 0) \to \tilde{X}, (z, 0) \nrightarrow Y \text{ off } \{\tilde{X}\}, \tilde{X}' \to Y) \end{aligned} \tag{5.23}$$

It is now easy to argue, say by the BKF inequality, that

$$\frac{d\chi_{L,\varepsilon}(\lambda; z)}{d\lambda} \leqslant \frac{|J|}{1 - \varepsilon\lambda\,|J|} (\chi_{L,\varepsilon}(\lambda; z))^2 \tag{5.24}$$

and we get (3.2) by then sending ε to zero.

The starting point for a complementary inequality is the observation that

$$\frac{d\chi_{L,\varepsilon}}{d\lambda}=\varepsilon^2 \sum_{\tilde{X},\tilde{X}',\,Y\in\Lambda_{L,\varepsilon}} \frac{J_{\tilde{X},\tilde{X}'}}{1-\varepsilon\lambda J_{\tilde{X},\tilde{X}'}} P^{\lambda}_{L,\varepsilon}(\mathscr{E}(\tilde{X},\tilde{X}';\{Y\};0)) \tag{5.25}$$

As in the preceding subsection, restrict $\tilde{X}$ to $(0,[1/\varepsilon]\,\varepsilon)+\Lambda_{L/3,\varepsilon}$ and force $\tilde{X}'=\tilde{X}+(e,0)$ with $J_{0,e}>0$. Now delocalize to obtain

$$\frac{d\chi_{L,\varepsilon}}{d\lambda}\geqslant\frac{aJ_{0,e}\varepsilon^2}{1-\varepsilon\lambda J_{0,e}} \sum_{\substack{X\in\Lambda_{L/3,\varepsilon}\\ Y\in\Lambda_{L,\varepsilon}}} P^{\lambda}_{L,\varepsilon}(\mathscr{E}_s(X,X';\{Y\};1;r)) \tag{5.26}$$

Factorizing the probability in (5.26) with the aid of Theorem 4.3 gives us a new lower bound on $d\chi_{L,\varepsilon}/d\lambda$ having the form of a single positive term ($\mathscr{L}$) less four corrections, which are the sums over X and Y of $Q_{B1,1,0}$, $Q_{B2,1,0}$, $Q_{I1,1,0}$, and $Q_{I2,1,0}$—all multiplied by $aJ_{0,e}/(1-\varepsilon J_{0,e})$.

Using arguments similar to those of the preceding subsection, we find that

$$\varepsilon^2\sum_{X,Y} Q_{B1,1,0} \leqslant 4\,|J|^2\,\nabla_{2L,\varepsilon}(\lambda;r-2R,r+2R,t)\,\chi_{L,\varepsilon}\chi_{2L,\varepsilon} \tag{5.27}$$

$$\varepsilon^2\sum_{X,Y} Q_{B2,1,0},\ \varepsilon^2\sum_{X,Y} Q_{I1,1,0} \leqslant 2\,|J|\,B_{2L,\varepsilon}(\lambda;r-R,r+R,t)\,\chi_{L,\varepsilon}\chi_{2L,\varepsilon} \tag{5.28}$$

and

$$\varepsilon^2\sum_{X,Y} Q_{I2,1,0}\leqslant A_{2L,\varepsilon}(\lambda;r,t)(\chi_{2L,\varepsilon})^2 \tag{5.29}$$

For the leading term, write $P^{\lambda}_{L,\varepsilon}(\mathbf{O}\to X,\mathbf{O}\nrightarrow Y)=P^{\lambda}_{L,\varepsilon}(\mathbf{O}\to X)-P^{\lambda}_{L,\varepsilon}(\mathbf{O}\to X,\mathbf{O}\to Y)$, apply the skeleton inequality (4.14), and make estimates of the type found earlier in this section to get that

$$\begin{aligned}\mathscr{L}&=\varepsilon^2\sum_{\substack{X\in\Lambda_{L/3,\varepsilon}\\ Y\in\Lambda_{L,\varepsilon}}} P^{\lambda}_{L,\varepsilon}(\mathbf{O}\to X,\mathbf{O}\nrightarrow Y)\,P(Z_1\to Y)\\ &\geqslant(\chi_{L/3,\varepsilon})^2-(2\,|J|\,\nabla_{2L,\varepsilon}(\lambda;r-R,r+R,t)+B_{2L,\varepsilon}(\lambda;r,r,t))\,\chi_{L,\varepsilon}\end{aligned} \tag{5.30}$$

Using the bounds (5.27)–(5.30) in the factorized quantity obtained from Theorem 4.3, taking the continuum limit, and using the monotonicity properties of the diagrams as in the preceding subsection, we find that

$$\frac{d\chi_L}{d\lambda} \geqslant aJ_{0,e}\{(\chi_{L/3})^2 - 2(|J|+1)\,\nabla(\lambda_c, r-R)\,\chi_L - 4(|J|+1)^2\,\nabla(\lambda_c, r-2R)(\chi_{2L})^2\} \tag{5.31}$$

for all $\lambda \leqslant \lambda_c$. Since $\chi(\lambda)$ is nonzero, nondecreasing and bounded from above on $[\lambda_0, \lambda_1]$ (recall that $\lambda_1 < \lambda_c$ by hypothesis), we have by a compactness argument similar to those which gave (5.20) and (5.21) that there is an L_0 such that whenever $L \geqslant L_0$ and $\lambda_0 \leqslant \lambda \leqslant \lambda_1$,

$$\chi_{L/3}(\lambda) \geqslant \tfrac{1}{3}\chi_L(\lambda) \qquad \text{and} \qquad \chi_{2L}(\lambda) \leqslant 3\chi_L(\lambda) \tag{5.32}$$

The desired inequality (3.3) now follows by taking r large—the term on the right-hand side of (5.31) which is linear in χ_L is negligible (in comparison with the other terms) when λ_0 is close to λ_c. ∎

APPENDIX. FACTORIZATION

Proof of Theorem 4.3. We give the proof for $n=1$, and then indicate how the argument changes for $n=2$.

Begin by conditioning on (or partitioning according to) the cluster, C, of the origin:

$$P(\mathscr{E}_s) = \sum_{A_0 \subset A_{L,\varepsilon}:\, X \in A_0} P(A_0 = C, A_0 \cap D = \varnothing)\, P(Z_1 \to D \text{ off } A_0) \tag{A.1}$$

Use (4.13) to get an upper bound on $P(Z_1 \to D \text{ off } A_0)$, and then sum over A_0 to get

$$P(\mathscr{E}_s) \leqslant P(\mathbf{O} \to X, \mathbf{O} \nrightarrow D)\, P(Z_1 \to D) \tag{A.2}$$

This proves the upper bound when $n=1$. If $n=2$, one further decomposes (A.1) according to the cluster (call it A_1) of Z_1 (in what is left of A_L after the removal of A_0), and then uses the upper bound from (4.13) twice—first for the connection from Z_2 to D, and then for the connection from Z_1.

The lower bound on $P(\mathscr{E}_s)$ follows by estimating the probability gained by replacing $P(Z_1 \to D \text{ off } A_0)$ with $P(Z_1 \to D)$ in the derivation of (A.2).

Now using the lower bound in (4.13)—which accounts for the entire difference between the two sides of (A.3)—and then performing the sums over A_0, we get

$$
\begin{aligned}
P(\mathscr{E}_s) \geqslant\ & P(\mathbf{O}\to X, \mathbf{O}\nrightarrow D)\, P(Z_1\to D) \\
& -\varepsilon\lambda \sum_{U',\,U} J_{U',\,U} P(\mathbf{O}\to X, \mathbf{O}\nrightarrow D, U\in\partial_H(C))\, P(Z_1\to U')\, P(U\to D) \\
& -\sum_U P(\mathbf{O}\to X, \mathbf{O}\nrightarrow D, U\in\partial_V(C))\, P(Z_1\to U)\, P(U\to D) \\
& -P(\mathbf{O}\to X, \mathbf{O}\to Z_1, \mathbf{O}\nrightarrow D)\, P(Z_1\to D) \qquad \text{(A.3)}
\end{aligned}
$$

In the first correction term, replace $P(\mathbf{O}\to X, \mathbf{O}\nrightarrow D, U\in\partial_H(C))$ by $P(\mathbf{O}\to X, \mathbf{O}\to U, \mathbf{O}\nrightarrow D)$ and then use the skeleton inequality (4.15) to bound this term from above by four terms which we describe instead of writing down. The first two terms together comprise the part of $Q_{B1,\,1,\,0}$ corresponding to the choice of $P(U\to D)$ in the last bracket in (4.17), the third term is the part of $Q_{B2,\,1,\,0}$ corresponding to the choice of $P(U\to D)$ in the bracket in (4.18), and the last term is zero since it involves the product $P(U\to X)\, P(Z_1\to U')$—recall that X preceeds Z_1 in time, and U and U' have the same time.

To recapture an important factor of ε, a more careful treatment of the term in (A.3) involving the vertical boundary of C is required. The key point to be realized is that for U to be in the vertical boundary of the origin's cluster, there must be some site U' already in that cluster which has U for a neighbor, and whose horizontal bond to U is open. Therefore,

$$
\begin{aligned}
& \sum_U P(\mathbf{O}\to X, \mathbf{O}\nrightarrow D, U\in\partial_V(C))\, P(Z_1\to U)\, P(U\to D) \\
& \quad\leqslant \sum_{U',\,U} P(\mathbf{O}\to X, \mathbf{O}\to U', \text{the } U' \text{ to } U \text{ bond is open}, \mathbf{O}\nrightarrow D) \\
& \qquad \times P(Z_1\to U)\, P(U\to D) \\
& \quad\leqslant \varepsilon\lambda \sum_{U',\,U} J_{U',\,U} P(\mathbf{O}\to X, \mathbf{O}\to U', \mathbf{O}\nrightarrow D)\, P(Z_1\to U)\, P(U\to D) \qquad \text{(A.4)}
\end{aligned}
$$

The right-hand side of (A.4) is analyzed as in the preceding paragraph, and bounded from above by the sum of four terms. To combine these terms more compactly with those found above, it is convenient to interchange U and U' here. Then the first two terms are just the part of $Q_{B1,\,1,\,0}$ corresponding to the choice of $P(U'\to D)$ in the last bracket in (4.17), the third term is the part of $Q_{B2,1,0}$ that comes from taking $P(U' \to D)$ in the bracket in (4.18), and again the last term is zero.

Finally, we use (4.15) to bound the last term in (A.3) from above by the sum of four terms. The first two combine to give us $Q_{I1,1,0}$, the third is $Q_{I2,1,0}$, and the fourth is zero because it contains $P(Z_1 \to X)$. This completes the proof of Theorem 4.3 in the case $n = 1$.

When $n = 2$, the argument is the same in spirit, although the details are messier. One first estimates the difference $P(Z_2 \to D) - P(Z_2 \to D$ off $A_0 \cup A_1)$. Half of the corrections resulting from this estimate involve connections that pass through A_0 and are treated just as above; these contibute to $Q_{B1,2,0}$, $Q_{B2,2,0}$, $Q_{I1,2,0}$, and $Q_{I2,2,0}$. The other half involve connections passing through A_1; these are analyzed in a similar fashion using (4.14) in place of (4.15), and make contributions to $Q_{B1,2,1}$, $Q_{B2,2,1}$, $Q_{B3,2,1}$, $Q_{V,2,1}$ and $Q_{I3,2,1}$. Let us comment on the presence of $Q_{B3,2,1}$, $Q_{V,2,1}$ and $Q_{I3,2,1}$, and on the absence of a "$Q_{I1,2,1}$" and a "$Q_{I2,2,1}$." In place of corrections involving $P(U \to X)\, P(Z_1 \to U')$, which we argued were zero, we now pick up negligible (but nonzero) terms involving $P(U \to Z_1)\, P(Z_2 \to U')$—these lead to $Q_{B3,2,1}$. There is an additional term appearing in the bound of $P(Z_1 \to D$ off A_0, $U \in \partial_V(C(Z_1)))$ since an alternative to the kind of decomposition we saw in (A.4) is that $U = Z_1$—this leads to $Q_{V,2,1}$. The entire correction stemming from the indicator function is $Q_{I3,2,1}$, which need not be further decomposed since it already contains the negligible factor $P(Z_1 \to Z_2, Z_1 \to D)$. Finally one has to compute corrections in the leading term due to the difference between $P(Z_1 \to D) - P(Z_1 \to D$ off $A_0)$, but this is exactly as in the case $n = 1$, except that there is now a factor of $P(Z_2 \to D)$ present. ∎

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